Parametric Oscillators

So far, have explored oscillators with varying frequencies, i.e.:

$$\ddot{x} + \omega^2(t)\, x = 0$$

For convenience:

$$\omega^2(t) = \omega_0^2 (1 + h \cos \Omega t), \qquad h < 1$$

→ $\Omega \ll \omega_0$ ⇒ adiabatic limit

⇒ adiabatic invariant theory.

→ $\Omega \gg \omega_0$ ⇒ ponderomotive limit

⇒ ponderomotive force, averaging, etc.

Now:

→ $\Omega \sim \omega_0$, especially $\Omega \sim 2\omega_0$

⇒ parametric instability.

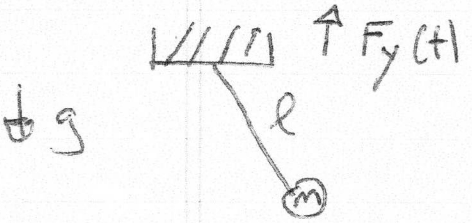
Plan:

- i) ideas.
- ii) theory of Mathieu equation.
- iii) method of variation of parameters.

i.) Parametric Instability – Ideas

→ consider pendulum with support acted on by vertical force

$\downarrow$ + → down



so $g \to g - F_y(t)/m$

$$\therefore 0 = \ddot{\theta} + \frac{g}{\ell}\theta \to \ddot{\theta} + \left(\frac{g}{\ell} - \frac{a(t)}{\ell}\right)\theta = 0$$

let $a(t) = a_0 \cos(\alpha t)$

$$\Rightarrow \ddot{\theta} + \omega_0^2\theta - \frac{a_0}{\ell}\cos(\alpha t)\,\theta = 0$$

$\left\{ \begin{array}{l} \text{see small } \phi \text{ limit, } \text{Pr. 12} \\ \alpha \not\gg \omega_0 \end{array} \right.$

$\therefore$ of Mathieu's equation genre, i.e.

$$\boxed{\ddot{x} + \omega_0^2(1 + a\cos(\gamma t))\,x = 0}$$

$\omega^2 = \omega^2(t)$, hence parametric oscillator

Parametric oscillator ↔ $\omega^2(t)$ periodic
oscillation of effective frequency

→ Some observations:

2) informal – consider what might happen?

for instability, observe can produce secularity if $\gamma \sim 2\omega_0$ via beat at fundamental

$$\underbrace{\ddot{x} + \omega_0^2 x}_{} + a \underbrace{\cos(\gamma t)}_{\downarrow\, \cos(2\omega_0 t)} \omega_0^2 \underbrace{x}_{\downarrow\, \sin(\omega_0 t)} = 0$$

$$\sim \sin(\omega_0 t)$$

resonant drive of fundamental oscillator

$\Rightarrow$ secularity $\rightarrow$ instability (why?)

$\therefore$ Solution of oscillator at ω_0 beats with parameter oscillation $\Rightarrow$ secularity

$\therefore$ Parametric resonance at/near $\gamma \sim 2\omega_0$ (twice fundamental)

Note: here $\omega^2 = \omega^2(t) \Rightarrow \partial L/\partial t \neq 0$ energy not conserved

$\Rightarrow$ work done on system (e.g. LGM oscillating pendulum support)

$\leftrightarrow$ source of energy for instability

? [What is relation of this to 3-mode parametric instability calculation (2004)] ?

Familiar example

- free asymmetric top : I_1, I_2, I_3
s/t

$$I_1 < I_2 < I_3$$

- Consider Poinsot construction

i.e. L/L fig : 5.1 (AG 117)

$$L_1^2 + L_2^2 + L_3^2 = L^2 = \text{const} \qquad \text{(sphere)}$$

$$\frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3} = E = \text{const} \qquad \text{(ellipsoid)}$$

$$\frac{d\underline{L}}{dt} = -\underline{\Omega} \times \underline{L} \, , \quad \frac{d\underline{L}}{dt} + \underline{\Omega} \times \underline{L} = \underline{\tau}$$

⇒ picture

⇒ Linear theory : i.e. axes 1,3 → stable
axes 2 → unstable.

~ parametrics

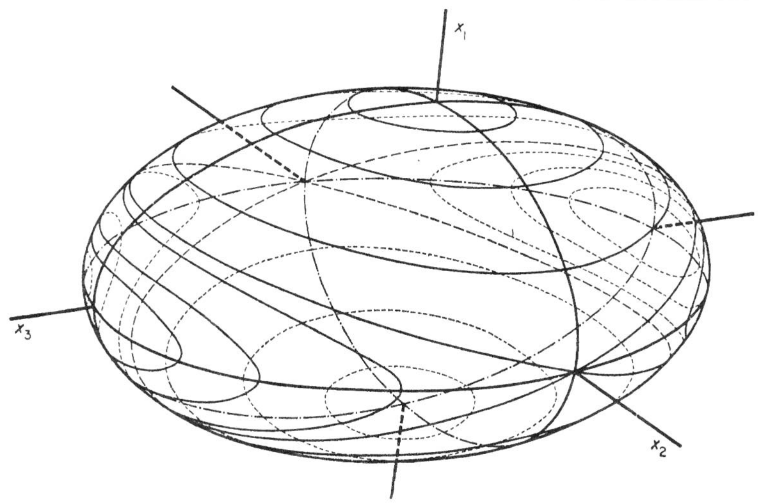


FIG. 51

(ii) formal (Floquet theory) $\Rightarrow$ (What type solution possible) $\left\{ \begin{array}{l} \text{what} \\ \text{Mathematics} \\ \text{Predicts} \end{array} \right.$

- $\omega(t)$ periodic, with period $T = 2\pi/\gamma$

$$\therefore \left\{ \begin{array}{l} \omega(t+T) = \omega(t) \\ \text{eqn. invariant under } t \to t+T \end{array} \right.$$

$\therefore$ if $x_1(t)$, $x_2(t)$ are 2 independent solutions of basic eqn.

$\Rightarrow$ $x_1(t)$, $x_2(t)$ must transform to linear combinations of themself upon $t \to t+T$ (linear eqn!)

and

can choose x_1, x_2 s/t $\left\{ \begin{array}{l} \text{diagonalize} \\ \text{transform.} \end{array} \right.$

$$\left\{ \begin{array}{l} x_1(t+T) = \mu_1 x_1(t) \\ x_2(t+T) = \mu_2 x_2(t) \end{array} \right.$$

(here "can choose" means can diagonalize transformation matrix)

$\Rightarrow$ most general functions having this property are:

$$\left\{ \begin{array}{l} x_1(t) = \mu_1^{t/T}\, \Pi_1(t) \\ x_2(t) = \mu_2^{t/T}\, \Pi_2(t) \end{array} \right. \qquad \left\{ \begin{array}{l} \text{where:} \\ \Pi_{1,2}(t+T) = \Pi(t) \end{array} \right.$$

- second, observe since linear equation
  $\Rightarrow$ Wronskian constant.

$$\dot{x}_2 x_1 - \dot{x}_1 x_2 = \text{const.}$$

to see:

$$x_2 \left(\ddot{x}_1 + \omega^2(t)\, x_1 \right) = 0 \qquad \Rightarrow \frac{d}{dt}\left(x_2 \dot{x}_1 - \dot{x}_2 x_1 \right) = 0$$
$$x_1 \left(\ddot{x}_2 + \omega^2(t)\, x_2 \right) = 0$$

but

$$W(x_1, x_2) = (\mu_1 \mu_2)^{-1}\, W\big(x_1(t+T), x_2(t+T)\big)$$

i.e. consider time translation by T

$\Rightarrow$ ∴ $\boxed{\mu_1 \mu_2 = 1}$

$$W(x_1, x_2) = \left(\mu_2^{t/T}\, \dot{\Pi}_2\right)\left(\mu_1^{t/T}\, \Pi_1\right) - \left(\mu_1^{t/T}\, \dot{\Pi}_1\right)\left(\mu_2^{t/T}\, \Pi_2\right)$$
$$= \left(e^{(\ln \mu_2) t/T}\, \dot{\Pi}_2\right) e^{(\ln \mu_1) t/T}\, \Pi_1 - \left(e^{(\ln \mu_1) t/T}\, \dot{\Pi}_1\right) e^{\ln \mu_2\, t/T}\, \Pi_2$$

- Can also observe:
  1) coeffs in oscillator real, so
     $x(t)$ an ~~~~ solution $\rightarrow x^*$ a solution
  $\Rightarrow$
  2) μ_1, μ_2 same as μ_1^*, μ_2^*
     i.e.

$$\text{I}: \begin{cases} \mu_1 = \mu_2^* \\ \mu_2 = \mu_1^* \end{cases} \quad \text{or} \quad \text{II}: \begin{cases} \mu_1 = \mu_1^* \\ \mu_2 = \mu_2^* \end{cases} \text{both real}$$

if I, $\mu_1 \mu_2 = 1 \Rightarrow \mu_1 = 1/\mu_1^*$, $\mu_2 = 1/\mu_2^*$ $\Rightarrow \underline{|\mu_1|^2 = |\mu_2|^2 = 1}$

(trivial)

non-trivial case

if II $\mu_1 \mu_2 = 1$; μ_1, μ_2 real $\Rightarrow$

$$\Rightarrow \boxed{x_1(t) = \mu^{t/T}\,\Pi_1(t), \quad x_2(t) = \mu^{-t/T}\,\Pi_2(t)}$$

i.e. 1 increasing, 1 decreasing } solution $\Rightarrow$ { parametric instability

[N.B. **Exponential**, not secular growth] !

$\Rightarrow$ "true" instability is possible

$\rightarrow$ Some Calculation (as basic structure of the solution established).

Consider Mathieu's eqn:

$$\ddot{x} + \omega_0^2\Big[1 + \underbrace{h}_{\text{strength}} \cos\big[(2\omega_0 + \underbrace{\epsilon}_{\text{mismatch}})t\big]\Big] x = 0$$

bounds on ϵ for instability ?

For solution, SHO $\Rightarrow$

$$x = a\cos(\omega_0 t) + b\sin(\omega_0 t)$$

so, in spirit of multiple-time-scale P.T. (i.e. $\omega^2(t)$ enters via $h \ll 1$ $\Rightarrow$ expect slow time scale variation of coefficients)

mismatch vs h $\rightarrow$ key competition.

$$x = \underbrace{a(t)}\cos[(\omega_0 + \epsilon/2)t] + \underbrace{b(t)}\sin[(\omega_0 + \epsilon/2)t]$$

coeffs become slowly varying

For more on multiple time scale perturbation theory → see Bender and Orszag, last chapter.

so, plugging in:

$$\begin{cases} \ddot{x} + \omega_0^2\left[1 + h\cos\left[(2\omega_0+\epsilon)t\right]\right] x = 0 \\ x = a(t)\cos\left[(\omega_0+\epsilon/2)t\right] + b(t)\sin\left[(\omega_0+\epsilon/2)t\right] \end{cases}$$

$$\ddot{x} = \ddot{\left[a(t)\cos\left[(\omega_0+\epsilon/2)t\right]\right]} + \ddot{\left[b(t)\sin\left((\omega_0+\epsilon/2)t\right)\right]}$$

$$= -(\omega_0+\epsilon/2)^2 a(t)\cos[\;]$$

$$- 2(\omega_0+\epsilon/2)\dot{a}(t)\sin[\;]$$

slow

$$+ \ddot{a}\cos[\;]$$

$$[\;] \equiv [(\omega_0+\epsilon/2)t]$$

$$- (\omega_0+\epsilon/2)^2 b(t)\sin[\;]$$

$$+ 2(\omega_0+\epsilon/2)\dot{b}(t)\sin[\;]$$

$$+ \ddot{b}(t)\sin[\;]$$

slow.

so, neglecting slow $\ddot{a}$, $\ddot{b}$:

$\Rightarrow$

$$-\left(\cancel{\omega_0^2} + \omega_0 \epsilon + \cancel{\epsilon^2/4}\right) a(t) \cos[\ \ ]$$

$$-2\dot{a}(t)\left(\omega_0 + \epsilon/2\right) \sin[\ \ ]$$

$$-\left(\cancel{\omega_0^2} + \omega_0 \epsilon + \cancel{\epsilon^2/4}\right) b(t) \sin[\ \ ]$$

$$+2\dot{b}(t)\left(\omega_0 + \cancel{\epsilon/2}\right) \cos[\ \ ]$$

$$+\omega_0^2 \left[a(t) \cos[\ \ ] + b(t) \sin[\ \ ]\right]$$

$$+\omega_0^2 h \cos\left[(2\omega_0 + \epsilon)t\right]\left[a(t)\cos[\ \ ] + b(t)\sin[\ \ ]\right] = 0$$

can simplify by:

- cancelling ω_0^2 terms
- drop $O(\epsilon^2)$.

so

so

$-\omega_0 \epsilon \left(a(t) \cos[\;] + b(t) \sin[\;] \right)$

$-2\dot{a}(\omega_0+\epsilon/2)\sin[\;] + 2\dot{b}(\omega_0+\epsilon/2)\cos[\;]$

$+\omega_0^2 h \cos((2\omega_0+\epsilon)t)\left[a(t)\cos[\;] + b(t)\sin[\;]\right]$

$=0$

Furthur, observe on ↑ :

$\cos[(\omega_0+\epsilon/2)t]\cos[(2\omega_0+\epsilon)t]$

$= \frac{1}{2}\cos[(\omega_0+\epsilon/2)t] + \frac{1}{2}\cos[3(\omega_0+\frac{\epsilon}{2})t]$

resonant

drives at $\sim\omega$

non-resonant

→ drives ω_0 osc. at $3\omega_0$

→ expect h.o. in h effect

→ drop

→ Resonant interaction is interesting one here..

$\Rightarrow$

$$-\omega_0\epsilon\left(a(t)\cos[\;] + b(t)\sin[\;]\right)$$

$$-2\dot{a}(\omega_0+\epsilon/2)\sin[\;] + 2\dot{b}(\omega_0+\epsilon/2)\cos[\;]$$

$$+\frac{\omega_0^2}{2}h\left[a(t)\cos[\;] - b(t)\sin[\;]\right]$$

$$=0$$

Regrouping coeffs. $\cos[\;]$, $\sin[\;]$;

$$\sin[\;]\left(-2\omega_0\dot{a} - b\omega_0\epsilon - \omega_0^2 h b/2\right)$$

$$+\cos[\;]\left(2\dot{b}\omega_0 - a\epsilon\omega_0 + \frac{1}{2}h\omega_0^2 a\right) = 0$$

$\Rightarrow$

$$(2\omega_0)\dot{a} + (\omega_0\epsilon)b + \left(\frac{\omega_0^2}{2}h\right)b = 0$$

$$(2\omega_0)\dot{b} - (\omega_0\epsilon)a + \left(\frac{\omega_0^2 h}{2}\right)a = 0$$

$\Rightarrow$

$$\boxed{\begin{cases}\dot{a} + (\epsilon/2)b + (\omega_0 h/4)b = 0\\ \dot{b} - (\epsilon/2)a + (\omega_0 h/4)a = 0\end{cases}}$$

Basic system of Eqns for Amplitude Variation

$a(t) = a_0 e^{st}$
$b(t) = b_0 e^{st}$

exponentially growing/damping solution

$\Rightarrow$

$$s\, a_0 + (\epsilon/2 + \omega_0 h/4)\, b_0 = 0$$

$$\left(-\frac{\epsilon}{2} + \frac{\omega_0 h}{4}\right) a_0 + s\, b_0 = 0$$

$$\therefore \quad \boxed{s^2 = \frac{\omega_0^2 h^2}{16} - \frac{\epsilon^2}{4} = \frac{1}{4}\left(\frac{\omega_0^2 h^2}{4} - \epsilon^2\right)}$$

$\Rightarrow$ Parametric instability criterion

Growthrate

Observe:

- instability for:

$$\epsilon^2 = (\gamma - 2\omega_0)^2 < \frac{\omega_0^2 h^2}{4}$$

$\omega_0 \to$ fundamental
$\gamma \to$ parametric variation freq.

amplitude of variation

$$h^2 > 4(\gamma - \omega_0)^2/\omega_0^2$$

i.e. sufficiently close to resonance $\Rightarrow$ growth (instability)

for $(\gamma - 2\omega_0)^2 > \omega_0^2 h^2/4$ $\to$ oscillation $\Rightarrow$ stable

- amplitude of $\omega_0^2(t)$ variation sets proximity threshold.

– more generally, can show when $\underset{\substack{\uparrow \\ \text{integer}}}{n}\gamma = 2\omega_0$ $\Rightarrow$ parametric resonance. Of course, higher n $\Rightarrow$ resonance region $\sim h^n$

– with friction, find threshold for instability:

i.e. $$(\gamma - 2\omega_0)^2 < \left[\left(\tfrac{1}{2} h\omega_0\right)^2 - 4\underset{\substack{\uparrow \\ \text{friction coeff.}}}{\alpha^2}\right]$$

i.e. P.I. growth must beat damping!
$\Rightarrow$ Friction raises required h.

– Pumping on swing:

$\rightarrow$ "pumping" $\rightarrow$ change of I

$$\ddot{\theta} + \frac{mgl\theta}{I(t)} = 0$$

$$I(t) = I_0 + \epsilon I_1(t)$$

$$\ddot{\theta} + \frac{g\theta}{\ell} + \frac{\epsilon g}{\ell}\frac{\Delta I}{I}(t)\theta = 0$$
$$+\alpha\dot{\theta}$$

need pump twice per cycle.

$\rightarrow$ Parametric instability is prototypical example of method of multiple scales P.T.

Note: – oscillation frequency ω_0

– growth:

$$s^2 = \frac{\omega_0^2 h^2}{16} - \frac{\epsilon^2}{4}$$

$$\Downarrow$$

$$(\gamma-2\omega)^2/4$$

Points: $s \ll \omega_0$.